International Journal of Theoretical Physics, Vol. 6, No. 2 (1972), pp. 123–131

Considerations on a Unified Field Theory

JACK COHN

Department of Physics and Astronomy, University of Oklahoma, Norman, Oklahoma

Received: 3 *August* 1971

Abstract

A geometrical formulation of gravitational and electromagnetic fields is presented for systems composed of point mass charged particles where the charge is small enough that electromagnetic radiation may be neglected. It is assumed that such charges produce a non-negligible contribution to the metric, and that their motion describes geodesics in the total metric which consists of that due to the charge itself and that due to the external environment of the charge. The above, together with several other assumptions yields the customary Einstein-Maxwell relations. It is demonstrated that this construction is not merely a re-statement of the Einstein-Maxwell theory in different terms.

1. *Introduction*

The problem of geometrizing the **E** and **H** fields of Maxwell is still unsolved. There have been many attempts at the construction of such a 'unified' field theory, but it is fair to say that none have won any particular acceptance. This is probably because no one has yet succeeded in giving a geometrical significance to the electromagnetic field. That is, the electromagnetic field has often been related (by postulate) to certain properties of certain spaces but this does not constitute their geometrical significance. For instance, in Weyl's gauge invariant theory (Weyl, 1918), the vector potential of the electromagnetic field is made responsible for the fact that 4-intervals have 'lengths' which are path dependent. At best, this only *relates* **E** and **H** to geometrical effects but, as Eddington (1960) has pointed out, does not give them the status of being a geometrical property of space-time. Again, in the later developments of Einstein (1945, 1955; Einstein & Staus, 1946) and Schroedinger (1947, 1948a, b), for example, we find ourselves even further removed from this goal, since, in these considerations, various requirements for Riemannian geometry are relaxed, yielding equations which, because of their *form*, are assumed to be those of Maxwell. This method of 'identification' is also to be found in the five-dimensional and projective theories (Bergmann, 1942). Finally, we should mention the rather different approach of Rainich (1925) and Wheeler (1961). In their considerations the **E** and **H** fields are related to geometry through Rainich's proof that they can (in certain cases) be expressed in terms of the Ricci tensor. Again, this constitutes no geometrical interpretation of these fields, but merely a *relation* to known geometrical quantities, such as the Ricci tensor.

The present work is concerned with the problem of constructing a formulation of general relativity which simultaneously gives geometrical significance to the electromagnetic field. In this formulation the **E** and **H** fields are not merely related to geometrical quantities but have a basic interpretation in terms of such properties. The theory to be presented does fall short in some ways, however, and is only to be considered as a first exploration along a rather different tack. It may be said that the ensuing theory is primarily (but not completely) the construction of a different language for the customary Einstein-Maxwell theory.

Further, it is to be noted that there may be many ways to satisfy the basic physical requirements of the theory to be described. The author has selected but one (the simplest to him) to consider in detail. The presentation really becomes, at one point then, the development of a model.

Finally, rather than introducing the various necessary assumptions all at once in axiomatic fashion, they shall be introduced more naturally in the course of the development. In this way, their motivation may be clarified.

2. *Construction of the Theory*

Before getting to the details of the theory, we must first consider what is to be meant by a geometrical theory of gravitational and electromagnetic fields.

We define a geometrical theory of gravitational and electromagnetic fields to satisfy the following principles:

(1) Point mass particles (charged or not) describe geodesics in space-time.

(2) The field equations describing space-time depend only on geometrically defined quantities, except for the appearance of certain constants.

The goal of the present work is to construct a formulation satisfying these two requirements, for a system composed of neutral and/or charged point particles where the charges are all of very small magnitude, which will also yield the customary Einstein-Maxwell description.

It will be assumed that the stress tensor due to the masses themselves vanishes outside the masses where it is singular.

We first momentarily remind the reader of the basic relations involved in the Einstein-Maxwell theory—they are:

$$G_{\mu\nu} = -\kappa T_{\mu\nu} \quad \text{(outside of mass singularities)} \tag{2.1a}$$

$$F_{\mu\nu,\lambda} + F_{\nu\lambda,\mu} + F_{\lambda\mu,\nu} = 0, \qquad F^{\mu\nu}_{\ ;\nu} = j^{\mu} \tag{2.1b}$$

and

$$\frac{\delta v^{\mu}}{\delta\tau}=\frac{e}{m_0 c^2}F^{\mu\nu}v_{\nu} \tag{2.2}$$

where commas denote ordinary differentiation, semi-colons denote covariant differentiation, $T_{\mu\nu}$ is the stress-energy momentum tensor due to just the electromagnetic field, $G_{\mu\nu}$ is the Einstein tensor, and v^{μ} denotes the 4-velocity of the particle.

Equation (2.2) is only correct for a particle with vanishingly small charge, so that radiation effects, which depend on e^2, can be neglected.

We begin our formulation with the assumption that a charged particle produces a non-negligible contribution to the metric, and that such a point mass charged particle moves along a geodesic in the *combined* metric due to the environment of the charge and the charge itself. Since the charge is assumed to be very small we shall assume that the total metric is a *linear* superposition of that due to the charge's environment and that due to the charge itself; that is, we assume that

$$g_{\mu\nu}=\overset{0}{g}_{\mu\nu}+\epsilon_{\mu\nu} \tag{2.3}$$

where $g_{\mu\nu}=g_{\nu\mu}$ is the total metric, and $\overset{0}{g}_{\mu\nu}=\overset{0}{g}_{\nu\mu}$ is the metric due to sources external to the charge, and $\epsilon_{\mu\nu}=\epsilon_{\nu\mu}$ is the metric produced by the charge itself.

From the above geodesic assumption we have

$$\frac{d^2x^{\mu}}{d\bar{\tau}^2}+\Gamma^{\mu}_{\alpha\beta}\frac{dx^{\alpha}}{d\bar{\tau}}\frac{dx^{\beta}}{d\bar{\tau}}=0 \tag{2.4}$$

where $d\bar{\tau}$ denotes arc-length due to the total metric and is given by $d\bar{\tau}^2=g_{\mu\nu}dx^{\mu}dx^{\nu}$, where $\Gamma^{\mu}_{\alpha\beta}$ is the symmetric connection given by

$$\Gamma^{\mu}_{\alpha\beta}=\tfrac{1}{2}g^{\mu\sigma}(g_{\alpha\sigma,\beta}+g_{\sigma\beta,\alpha}-g_{\alpha\beta,\sigma})\equiv g^{\mu\sigma}[\alpha\beta,\sigma]^{g} \tag{2.5}$$

and where Greek indices take values from 1 to 4, and Latin indices go from 1 to 3.

Using the decomposition of the metric [equation (2.3)], we have

$$\Gamma^{\mu}_{\alpha\beta}=\overset{0}{\Gamma}{}^{\mu}_{\alpha\beta}+\Delta^{\mu}_{\alpha\beta} \tag{2.6}$$

where

$$\overset{0}{\Gamma}{}^{\mu}_{\alpha\beta}=\overset{0}{g}{}^{\mu\sigma}[\alpha\beta,\sigma]^{\overset{0}{g}} \tag{2.7}$$

Since $\Gamma^{\mu}_{\alpha\beta}$ and $\overset{0}{\Gamma}{}^{\mu}_{\alpha\beta}$ are both symmetric connections, it follows that $\Delta^{\mu}_{\alpha\beta}\equiv\Gamma^{\mu}_{\alpha\beta}-\overset{0}{\Gamma}{}^{\mu}_{\alpha\beta}$ is a tensor of the indicated rank. For the moment we shall not need the explicit form of $\Delta^{\mu}_{\alpha\beta}$.

From equation (2.4) we now have

$$\frac{d^2 x^\mu}{d\bar{\tau}^2} + \overset{0}{\Gamma}{}^\mu_{\alpha\beta} \frac{dx^\alpha}{d\bar{\tau}} \frac{dx^\beta}{d\bar{\tau}} = -\Delta^\mu_{\alpha\beta} \frac{dx^\alpha}{d\bar{\tau}} \frac{dx^\beta}{d\bar{\tau}} \tag{2.8}$$

Letting $d\tau$ denote the arc-length associated with the background metric (i.e. $d\tau^2 = \overset{0}{g}_{\mu\nu} dx^\mu dx^\nu$) we can write equation (2.8) as

$$\frac{d^2 x^\mu}{d\tau^2} + \overset{0}{\Gamma}{}^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = -\left(\frac{d\bar{\tau}}{d\tau}\right)^2 \frac{d^2 \tau}{d\bar{\tau}^2} \frac{dx^\mu}{d\tau} - \Delta^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} \tag{2.9}$$

where we have used the relations

$$\frac{dx^\mu}{d\bar{\tau}} = \frac{dx^\mu}{d\tau} \frac{d\tau}{d\bar{\tau}}, \qquad \frac{d^2 x^\mu}{d\bar{\tau}^2} = \left(\frac{d\tau}{d\bar{\tau}}\right)^2 \frac{d^2 x^\mu}{d\tau^2} + \frac{dx^\mu}{d\tau} \frac{d^2 \tau}{d\bar{\tau}^2} \tag{2.10}$$

along the particle path in space-time.

Further, from the definition of $d\tau$ and $d\bar{\tau}$, we have

$$\left(\frac{d\bar{\tau}}{d\tau}\right)^2 = 1 + \epsilon_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \tag{2.11}$$

Now we can obtain an equation of the form of equation (2.2) by assuming that $\epsilon_{44} = 0$ in some frame for which all $dx^i/d\tau = 0$. (Note that this does not imply that $\epsilon_{44} = 0$ in every rest frame of the charge.)

Equation (2.11) then gives

$$\frac{d\bar{\tau}}{d\tau} = 1 \tag{2.12}$$

along the particle path in space-time.

We see then that segments along the particle path have the same 4-length in either metric; $g_{\alpha\beta}$ or $\overset{0}{g}_{\alpha\beta}$. Of course, this will not be true in general for segments not on the particle path.

We note that equation (2.12) could have been extracted from equation (2.11) by requiring, instead, that $\epsilon_{\mu\nu}$ be anti-symmetric. This, however, would have prevented us from assuming that both $g_{\alpha\beta}$ and $\overset{0}{g}_{\alpha\beta}$ were symmetric. The associated connections would then not both be symmetric and could not both be identified with the Christoffel symbols. One would then have to postulate the dependence of such connections on the metric. We have avoided this assumption at the expense of the simpler (to us) postulate following equation (2.11).

Returning to equation (2.12), we obtain

$$\frac{d^2 \tau}{d\bar{\tau}^2} = 0 \tag{2.13}$$

along the particle path.

Therefore, equation (2.9) takes the form

$$\frac{\delta v^\mu}{\delta \tau} = -\Delta^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} \tag{2.14}$$

Comparing equations (2.4) and (2.14) we see that a path which is a geodesic in the total metric, $g_{\alpha\beta}$, in general is not a geodesic in the background metric, $\overset{0}{g}_{\alpha\beta}$.

We now see how to retrieve equation (2.2) from our formalism; we shall let $-\Delta^{\mu}_{\alpha\beta}v^{\alpha}$ essentially play the role of the Maxwell field tensor, and shall re-interpret the customary relation (2.2) to hold in the background metric, not in the total metric. For the most part, what is regarded as *the* metric in the customary formulation will be replaced by the background metric in our formulation.

We therefore assume that, at the particle event

$$-\Delta^{\mu}_{\alpha\beta}v^{\alpha} = \frac{e}{m_0 c^2} F_{\beta}{}^{\mu} \tag{2.15}$$

where $F^{\mu\nu}$ is the Maxwell field tensor, and where m_0 is the rest mass of the particle. *This relation really amounts to an interpretation of the field tensor in terms of geometrical quantities.*

We assume then that the following Maxwell equations are satisfied

$$(-\Delta^{\mu\beta}_{\alpha}v^{\alpha})_{;\beta} = \frac{e}{m_0 c^2} j^{\mu} \tag{2.16}$$

and

$$(\Delta_{\mu\alpha\nu}v^{\alpha})_{,\sigma} + (\Delta_{\nu\alpha\sigma}v^{\alpha})_{,\mu} + (\Delta_{\sigma\alpha\mu}v^{\alpha})_{,\nu} = 0 \tag{2.17}$$

Therefore, equation (2.14) takes the form

$$\frac{\delta v^{\mu}}{\delta\tau} = \frac{e}{m_0 c^2} F_{\beta}{}^{\mu} v^{\beta} \tag{2.18}$$

and we have recovered the customary formalism except for the field equations.

It is important to note here the very different interpretation being given to the field tensor, $F_{\mu\nu}$. In the customary formulation the field tensor is composed of electric and magnetic fields acting on a charged test particle, these fields being produced by other charges. In the present formulation, these **E** and **H** fields acting on a charge are considered as manifestations of metrical properties produced by that same charge, these metrical properties being, however, in part produced by the other charges.

Before discussing the field equations, we shall consider the present interpretation in more detail.

From the discussion preceding equation (2.15) we have that each charge acts as a source for all the others' production of $\Delta^{\mu}_{\alpha\beta}v^{\alpha}$. This is a Machian type of mechanism which doesn't seem to be found in the customary formulation of general relativity. We notice also that $\Delta^{\mu}_{\alpha\beta}v^{\alpha}$ only has relevance at the test particle event. This means that the electromagnetic field only need be considered at the location of the detecting charge. For very small e, as we are requiring, where electromagnetic radiation effects may be neglected, this interpretation of the field is valid, as considerations in the Wheeler-Feynman absorber theory of radiation shows (Wheeler & Feynman, 1945, 1949). Finally, we note the introduction of the non-geometrical quantity, e, in our equations. This, however, seems no worse than the introduction of the mass, m, in conventional general relativity theory. Neither theory completely explains its sources. In the conventional theory one can define m as a suitable integral of the pseudo-energy momentum tensor; in our case, we could define e as an integral of $\Delta^{\mu}_{\alpha\beta} v^{\alpha}$ over a suitable closed surface.

3. *Solution for $\Delta^{\mu}_{\alpha\beta}$ and $\epsilon_{\mu\nu}$*

The quantities, $\Delta^{\mu}_{\alpha\beta}$, must depend on the 4-velocity of the test charge in such a way that $\Delta^{\mu}_{\alpha\beta} v^{\alpha}$ is independent of $\{v^{\sigma}\}$, by assumption. Thus, equations (2.16) and (2.17) are to be regarded as ordinary differential equations in the quantities $\Delta^{\mu}_{\alpha\beta} v^{\alpha}$.

After solving these equations for these quantities we are still left with the problem of finding the $\Delta^{\mu}_{\alpha\beta}$ themselves. The expression of the $\Delta^{\mu}_{\alpha\beta}$ in terms of the $\Delta^{\mu}_{\alpha\beta} v^{\alpha}$ (or $F_{\beta}{}^{\mu}$) is non-unique, and we now consider the simplest (to the author) such solution, which may be said to comprise then a special model.

We take the $\Delta^{\mu}_{\alpha\beta}$ to be given by the relation (at the particle event only)

$$-\Delta^{\mu}_{\alpha\beta} = \frac{e}{m_0 c^2} \{F_{\beta}{}^{\mu} v_{\alpha} + F_{\alpha}{}^{\mu} v_{\beta} - F_{\sigma}{}^{\mu} v^{\sigma} v_{\alpha} v_{\beta}\} \tag{3.1}$$

This expression yields the necessary relation

$$-\Delta^{\mu}_{\alpha\beta} v^{\alpha} = \frac{e}{m_0 c^2} F_{\beta}{}^{\mu}$$

and preserves the symmetry of $\Delta^{\mu}_{\alpha\beta}$ in the indices α and β. (Note that another solution is given by adding the term, $\Lambda^{\mu} a_{\alpha} a_{\beta}$, where Λ^{μ} is an arbitrary 4-vector, and a_{α} is the 4-acceleration.)

We now consider equation (3.1) in the 'flat-space' limit, where it is possible to make inferences about the metric, $\epsilon_{\alpha\beta}$.

By the flat-space limit we mean that the test charge under consideration is very far from all other masses, so that $\overset{0}{g}_{\alpha\beta} \simeq \eta_{\alpha\beta} = \mathrm{diag}(-1,-1,-1,1)$. Here, even though the $\epsilon_{\alpha\beta}$ are small compared with the $\eta_{\alpha\beta}$ (except when $\alpha \neq \beta$) it is still possible that $\overset{0}{\Gamma}{}^{\mu}_{\alpha\beta}$ is negligible in comparison with quantities containing derivatives of the $\epsilon_{\mu\nu}$.

To first order in the small quantities $\epsilon_{\mu\nu}$, we have

$$\Delta^{\mu}_{\alpha\beta} \equiv \Gamma^{\mu}_{\alpha\beta} - \overset{0}{\Gamma}{}^{\mu}_{\alpha\beta} \simeq \eta^{\mu\sigma}[\alpha\beta, \sigma]^{\epsilon} - \eta^{\nu\mu} \epsilon_{\lambda\nu} \overset{0}{\Gamma}{}^{\lambda}_{\alpha\beta} \tag{3.2}$$

Thus, in the flat-space limit we have

$$\Delta^{\mu}_{\alpha\beta} \simeq \tfrac{1}{2}\eta^{\mu\mu}(\epsilon_{\alpha\mu,\beta} + \epsilon_{\beta\mu,\alpha} - \epsilon_{\alpha\beta,\mu}) \quad \text{(no sum on } \mu) \tag{3.3}$$

From this we directly obtain the relations

$$\Delta^i_{j\beta} + \Delta^j_{i\beta} = -\epsilon_{ij,\,\beta}$$
$$\Delta^4_{4\beta} = \tfrac{1}{2}\epsilon_{44,\,\beta} \tag{3.4}$$
$$\Delta^i_{4\beta} - \Delta^4_{i\beta} = -\epsilon_{i4,\,\beta}$$

Using these relations and equation (3.1) we can now find the proper time derivatives of the $\epsilon_{\alpha\beta}$ along the particle path, as follows

$$\frac{d\epsilon_{ij}}{d\tau} = \epsilon_{ij,\,\beta}v^\beta = -(\Delta^i_{j\beta} + \Delta^j_{i\beta})v^\beta = 0 \tag{3.5}$$

Similarly,

$$\frac{d\epsilon_{44}}{d\tau} = \epsilon_{44,\,\beta}v^\beta = 2\Delta^4_{4\beta}v\beta = 0 \tag{3.6}$$

and

$$\frac{d\epsilon_{i4}}{d\tau} = \epsilon_{i4,\,\beta}v^\beta = -(\Delta^i_{4\beta} - \Delta^4_{i\beta})v^\beta = \frac{2e}{m_0 c^2}F_4{}^i \tag{3.7}$$

Thus, only the space-time components of $\epsilon_{\alpha\beta}$ change along the particle path. Therefore, for particles which have come into the field from outside of it, the ϵ_{ij} and ϵ_{44} have values determined by zero field. Apparently, these components of $\epsilon_{\alpha\beta}$ only depend on conserved kinematical quantities.

It is also of interest to calculate the $\epsilon_{\alpha\beta,\,\lambda}$ when $\mathbf{v} = 0$ (when $v^\mu = (0,0,0,1)$. Such quantities will be denoted as $\epsilon^{(0)}_{\alpha\beta,\lambda}$. We obtain the following relations

$$\epsilon^{(0)}_{ij,\,\beta} = 0, \qquad \epsilon^{(0)}_{44,\,4} = 0, \qquad \epsilon^{(0)}_{i4,\,j} = \frac{e}{m_0 c^2}F_j{}^i$$
$$\epsilon^{(0)}_{i4,\,4} = -\frac{2e}{m_0 c^2}F_i{}^4, \qquad \epsilon^{(0)}_{44,\,i} = -\frac{2e}{m_0 c^2}F_i{}^4 \tag{3.8}$$

We note that ϵ_{44} behaves locally in proportion to the potential at the location of the charge.

4. *Measureable Consequences*

So far, it appears like the present formulation is merely a different language for the customary Einstein-Maxwell theory. If this were true, we could not expect any testable predictions differing from those of the customary formulation to arise from our theory. It turns out, however, that the present formulation is more than just a rephrasing of the Einstein-Maxwell theory. We can see that this must be so since, for instance, in the flat-space limit where the customary theory predicts that the (entire) metric is essentially diagonal, we have only that the background metric is diagonal. The metric, though vanishingly small, may still be detectable by simultaneously relating it to geometry and the electromagnetic field.

Quantitatively, we proceed as follows:
We have, from equation (2.6), the exact relation

$$\Gamma^{\alpha}_{\alpha\beta} = \overset{0}{\Gamma}{}^{\alpha}_{\alpha\beta} + \Delta^{\alpha}_{\alpha\beta} \tag{4.1}$$

and therefore,

$$\Delta^{\alpha}_{\alpha\beta} = \left[\ln \sqrt{\left(\frac{g}{\overset{0}{g}}\right)}\right]_{,\beta} \tag{4.2}$$

where g and $\overset{0}{g}$ denote the determinants of the metric $g_{\alpha\beta}$ and $\overset{0}{g}_{\alpha\beta}$ respectively.
In the flat-space limit we then have

$$\Delta^{\alpha}_{\alpha\beta} \simeq (\ln \sqrt{g})_{,\beta} \tag{4.3}$$

Therefore, using equation (3.4) we have

$$\Delta^{\alpha}_{\alpha\beta} = -\tfrac{1}{2}\epsilon_{ii,\,\beta} + \tfrac{1}{2}\epsilon_{44,\,\beta} = (\ln \sqrt{g})_{,\beta} \tag{4.4}$$

Finally, utilizing equation (3.8), for the case when the charge is momentarily at rest we have

$$-\frac{e}{m_0 c^2} F_i{}^4 = (\ln \sqrt{g})_{,i} \tag{4.5}$$

This relation is testable, in principle, for it claims a relation between geometry $[(\ln \sqrt{g})_{,i}]$, in a co-ordinate system where $\overset{0}{\Gamma}{}^{\mu}_{\alpha\beta} \doteq 0$, and a measurable field, $F_i{}^4$. Of course, at the location of the charge things actually get complicated because the mass of the charge will also contribute to the metric there. However, this contribution might be accounted for, and subtracted off. At any rate, what is being considered here is a matter of principle. Thus, the present formulation is more than merely a re-statement of the customary interpretation.

5. *The Field Equations*

Our final assumption concerns the field equations.

According to requirement (2.2) these should depend only on geometrical quantities. We shall also invoke the assumption, once again, that quantities depending on the (entire) metric in the customary formulation shall depend only on the background metric in our formulation.

Thus we assume that the field equations are given by replacing, in the customary expression, the $F^{\mu\nu}$ entering $T^{\mu\nu}$ by the $\Delta^{\mu}_{\alpha\beta} v^{\alpha}$ as given in equation (2.15); we have then

$$\overset{0}{G}{}^{\mu\nu} = -\kappa \left(\frac{m_0 c^2}{e}\right)^2 . \{\Delta^{\mu}_{\alpha\sigma} v^{\alpha} \Delta^{\sigma\nu}_{\alpha} v^{\alpha} + \tfrac{1}{4} g^{\mu\nu} \Delta_{\alpha\lambda\beta} v^{\lambda} \Delta^{\alpha\beta}_{\lambda} v^{\lambda}\} \tag{5.1}$$

where the quantity on the left-hand side is the Einstein tensor in the background metric.

If the charge on the point particle is zero, the above right-hand side vanishes, and we recover the customary Einstein relation for the vacuum, (that is, where the matter tensor vanishes).

Presumably, the right-hand side of equation (5.1) above describes some condition of balance for the charge, but it seems to have no known *overall* geometrical interpretation.

It would have been tempting to have postulated, instead, an equation like, $G_{\mu\nu} = 0$, at the particle event, where $G_{\mu\nu}$ is the Einstein tensor in the entire metric $g_{\alpha\beta}$. If such equations are, however, to be equivalent to the relations $\overset{0}{G}_{\mu\nu} = -\kappa T_{\mu\nu}$, then we would have to postulate certain additional relations between the $\Delta^{\mu}_{\alpha\beta} v^{\alpha}$, which would then overdetermine them.

Equations (5.1), (2.16), (2.17) and (2.18) thus comprise the basic relations of our formulation.

References

Bergmann, P. G. (1942). *Introduction to the Theory of Relativity*, Chapter 17. Prentice-Hall, New York.
Eddington, A. S. (1960). *The Mathematical Theory of Relativity*, Chapter 7. Cambridge University Press.
Einstein, A. (1945). *Annals of Mathematics*, **46**, 578.
Einstein, A. (1955). *The Meaning of Relativity*, Appendix II. Princeton University Press, Princeton, New Jersey.
Einstein, A. and Straus, G. (1946). *Annals of Mathematics*, **47**, 731.
Rainich, G. Y. (1925). *Trans. American Mathematical Society*, **27**, 106.
Schroedinger, E. (1947). *Proceedings of the Royal Irish Academy*, **51**, 163.
Schroedinger, E. (1948a). *Proceedings of the Royal Irish Academy*, **52**, 1.
Schroedinger, E. (1948b). *Proceedings of the Royal Irish Academy*, **51**, 205.
Weyl, H. (1918). *Sitzungsberichte*. Berlin.
Wheeler, J. A. (1961). *Geometrodynamics*. Academic Press, New York.
Wheeler, J. A. and Feynman, R. P. (1945). *Review of Modern Physics*, **17**, 157.
Wheeler, J. A. and Feynman, R. P. (1949). *Review of Modern Physics*, **21**, 425.